JEE Main 2023 (Memory based)

24 January 2023 - Shift 2

Answer & Solutions

MATHEMATICS

1. $\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}} dx$ is equal to:

A. π

B. 2π

C. -2π

D. 3π

Answer (B)

Sol.

$$I = \int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{3\times\sqrt{1-\frac{4x^2}{9}}} dx$$

Let $\frac{2x}{3} = t$

$$I = 16\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{\frac{3}{2}}{\sqrt{1-t^2}} dt$$

$$I = 24\sin^{-1} t \,\Big|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$I = 24\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = 2\pi$$

2. $\left(\frac{1+cos\left(\frac{2\pi}{9}\right)+i\,sin\left(\frac{2\pi}{9}\right)}{1+cos\left(\frac{2\pi}{9}\right)-i\,sin\left(\frac{2\pi}{9}\right)}\right)^3$ is equal to:

A. $-\frac{1}{2}+\frac{i\sqrt{3}}{2}$

B. $-\frac{1}{2}-\frac{i\sqrt{3}}{2}$

C. $\frac{1}{2}-\frac{i\sqrt{3}}{2}$

D. $\frac{1}{2}+\frac{i\sqrt{3}}{2}$

Answer (A)

Sol.

$$\left(\frac{1+cos\left(\frac{2\pi}{9}\right)+i\,sin\left(\frac{2\pi}{9}\right)}{1+cos\left(\frac{2\pi}{9}\right)-i\,sin\left(\frac{2\pi}{9}\right)}\right)^3 = \left(\frac{2\cos\left(\frac{\pi}{9}\right)\left(\cos\frac{\pi}{9}+i\sin\left(\frac{\pi}{9}\right)\right)}{2\cos\left(\frac{\pi}{9}\right)\left(\cos\frac{\pi}{9}-i\sin\left(\frac{\pi}{9}\right)\right)}\right)^3$$

$$= \left(e^{\frac{i2\pi}{9}}\right)^3$$

$$= e^{\frac{i2\pi}{3}}$$

$$= -\frac{1}{2}+\frac{i\sqrt{3}}{2}$$

3. If $\vec{a} = \hat{\imath} + 2\hat{\jmath} + m\hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + m\hat{k}$ if $\vec{a}$ & $\vec{b}$ are perpendicular to each other then m equals:

A. $\pm\sqrt{2}$
B. $\pm\sqrt{3}$
C. ± 2
D. $\pm\sqrt{5}$

Answer (B)

Sol.

If $\vec{a}$ & $\vec{b}$ are perpendicular to each other then $\vec{a} \cdot \vec{b} = 0$
$\Rightarrow (\hat{\imath} + 2\hat{\jmath} + m\hat{k}) \cdot (\hat{\imath} - 2\hat{\jmath} + m\hat{k}) = 0$
$\Rightarrow 1 - 4 + m^2 = 0$
$\Rightarrow m^2 = 3$
$\Rightarrow m = \pm\sqrt{3}$

4. The sum of the coefficients of first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^n$ is 376. The coefficient of x^4 is equal to:

A. 695
B. 410
C. 405
D. 395

Answer (C)

Sol.

The first three of $\left(x - \frac{3}{x^2}\right)^n = {}^nC_0 x^n - {}^nC_1 \cdot 3x^{n-3} + {}^nC_2 \cdot 9x^{n-6}$
${}^nC_0 - {}^nC_1 \cdot 3 + {}^nC_2 \cdot 9 = 376$
$\Rightarrow 1 - 3n + \frac{n(n-1)}{2} \times 9 = 376$
$\Rightarrow 3n^2 - 5n - 250 = 0$
$\Rightarrow n = 10$
Now,
$T_{r+1} = {}^{10}C_r x^{10-r} \cdot \left(-\frac{3}{x^2}\right)^r$
$\Rightarrow {}^{10}C_r (-3)^r \cdot x^{10-3r}$
$\Rightarrow 10 - r - 2r = 4$
$\Rightarrow r = 2$
Coefficient of $x^4 = {}^{10}C_2 (-3)^2$
$= 45 \times 9 = 405$

5. Let $A = \{a, b, c, d\}$ and a relation $A \rightarrow A$ be $R = \{(a,b), (b, d), (b, c), (b, a)\}$ then minimum number of elements required to make R equivalent is:

A. 7
B. 10
C. 12
D. 14

Answer (C)

Sol.

Adding $(a,\ a)$, $(b,\ b)$, $(c,\ c)$, $(d,\ d)$ makes it reflexive.

Adding $(d,\ b)$ and $(c,\ b)$ makes it symmetric.

And adding (a, d), (a, c) makes it transitive.

So further $(d,\ a)$ & $(c,\ a)$ to be added to maintain symmetricity of relation.

further $(c,\ d)$ & $(d,\ c)$ also be added.

Hence total of 12 elements to be added to mole equivalence.

6. 3 urns A, B, C contain 4 red, 6 black; 5 red, 5 black; λ red, 4 black balls. A ball is drawn and found to be red. If probability that ball was drawn from urn C is 0.4, then the square of side of equilateral triangle inscribed in parabola $y^2 = \lambda x$ with one vertex at vertex of parabola is

A. 144
B. 432
C. 368
D. 284

Answer (B)

Sol.

P(Red ball from urn C) $= \dfrac{\frac{1}{3}.\frac{\lambda}{\lambda+4}}{\frac{1}{3}.\frac{4}{10}+\frac{1}{3}.\frac{5}{10}+\frac{1}{3}.\frac{\lambda}{\lambda+4}} = \dfrac{4}{10}$

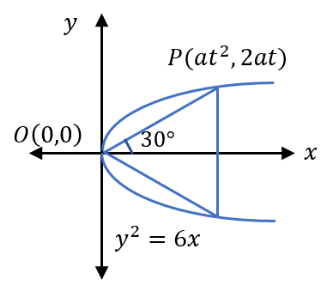
$\Rightarrow\ \lambda = 6$

$m_{op} = \frac{1}{\sqrt{3}}$

$\frac{2}{t} = \frac{1}{\sqrt{3}}$

Length of side $= 4at = 12\sqrt{3}$ units

Square of side length $= 432$



7. Total number of numbers formed using digits $3,\ 5,\ 6,\ 7,\ 8$ (without repetition) which are greater than 7000, is equal to:

A. 148
B. 168
C. 144
D. 124

Answer (B)

Sol.

Number using all the 5 digits $= 5! = 120$
Number using 4 digits
Case I:
When 7 is fixed at $1000's$ place
7 ___ ___ ___ $= 24$ ways
Case II:
When 8 is fixed at $1000's$ place

$8\_\_\_\ \_\_\_\ \_\_\_ = 24$ ways
Total number $= 120 + 24 + 24$
$= 168$

8. A 5×5 matrix whose each entry is either 0 or 1, is such that sum of entries of each column as well as each row is 1. Number of such matrices is :

A. 30
B. 60
C. 90
D. 120

Answer (D)

Sol.

In first column, 1 can be placed in any of the 5 places $= 5$
In second column, 1 can be placed in any of the 4 places $= 4$
In third column, 1 can be placed in any of the 3 places $= 3$
In fourth column, 1 can be placed in any of the 2 places $= 2$
In fifth column, 1 can be placed in only 1 place $= 1$

Total $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

9. If the shortest distance between the lines $\frac{x-\sqrt{6}}{1} = \frac{y+\sqrt{6}}{2} = \frac{z-\sqrt{6}}{3}$ and $\frac{x-\lambda}{3} = \frac{y-2\sqrt{6}}{4} = \frac{z-3\sqrt{6}}{5}$ is 6, then the sum of squares of all possible values of λ is equal to:

A. 1024
B. 732
C. 416
D. 312

Answer (B)

Sol.

$\overrightarrow{b_1} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$
$\overrightarrow{b_2} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$
$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(\lambda - \sqrt{6}\right)\hat{\imath} + 3\sqrt{6}\hat{\jmath} + 2\sqrt{6}\hat{k}$
$\overrightarrow{b_1} \times \overrightarrow{b_2} = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$
$d = \left|\frac{(\overrightarrow{a_2}-\overrightarrow{a_1})\cdot(\overrightarrow{b_1}\times\overrightarrow{b_2})}{|\overrightarrow{b_1}\times\overrightarrow{b_2}|}\right| \Rightarrow 6 = \left|\frac{-2\left(\lambda-\sqrt{6}\right)+12\sqrt{6}-4\sqrt{6}}{\sqrt{24}}\right|$
$\Rightarrow \lambda = 11\sqrt{6},\ -\sqrt{6}$
$\Rightarrow \lambda_1^2 + \lambda_2^2 = 732$

10. The proposition $\sim\left(p \wedge (p \rightarrow \sim q)\right)$ is equivalent to :

A. $p \wedge (p \vee q)$
B. $\sim p \vee q$
C. $p \vee q$
D. $\sim p \wedge q$

Answer (B)

Sol.

$$\sim(p \wedge (p \rightarrow \sim q))$$
$$= \sim(p \wedge (\sim p \vee \sim q))$$
$$= \sim(f \vee (p \wedge \sim q))$$
$$= \sim(p \wedge \sim q)$$
$$= \sim p \vee q$$

11. $({}^{30}C_1)^2 + 2 \cdot ({}^{30}C_2)^2 + \cdots + 30 \cdot ({}^{30}C_{30})^2 = \frac{\alpha \cdot 60!}{(30!)^2} - 1$ then α is:

A. 12
B. 15
C. 10
D. 13

Answer (B)

Sol.

$$\sum_{r=1}^{30} r \cdot {}^{30}C_r \cdot {}^{30}C_r = 30 \sum_{r=1}^{30} {}^{29}C_{r-1} \cdot {}^{30}C_{30-r}$$

$= 30 \cdot$ coefficient of x^{29} in $(1+x)^{29} \cdot (1+x)^{30}$

$$= 30 \cdot {}^{59}C_{29}$$

$$= 30 \cdot \frac{59!}{29!30!}$$

$$\Rightarrow \frac{900 \cdot 59!}{(30!)^2}$$

$$\Rightarrow \frac{15 \cdot 60!}{(30!)^2}$$

$$\Rightarrow \alpha = 15$$

12. If $\lim_{x \to \alpha} |[x-5] - [2x+2]| = 0$ then:

A. $\alpha \in (-7.5, -6.5)$
B. $\alpha \in [-7.5, -6.5)$
C. $\alpha \in (-7.5, -6.5]$
D. $\alpha \in [-7.5, -6.5]$

Answer (B)

Sol.

$$\lim_{x \to \alpha} |[x-5] - [2x+2]|$$

$$\lim_{x \to \alpha} |[x] - 5 - [2x] - 2|$$

$$\Rightarrow \lim_{x \to \alpha} |[x] - [2x] - 7|$$

$$\lim_{x \to -7.5^+} |[x] - [2x] - 7|$$

$$\Rightarrow |-8 + 15 - 7| = 0$$

$At\ x = -7.5, |-8+15-7| = 0$

$\lim_{x \to -6.5^-} |[x] - [2x] - 7|$

$\Rightarrow |-7+14-7| = 0$

$At\ x = -6.5, |[x] - [2x] - 7|$

$\Rightarrow |-7+13-7| \neq 0$

$\therefore \alpha \in [-7.5, -6.5)$

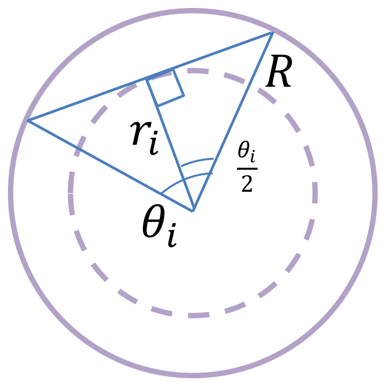
13. The locus of the mid points of chords of the circle $(x-4)^2 + (y-5)^2 = 4$ which subtends angle θ_i at the centre of this circle is a circle of radius r_i. If $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{2\pi}{3}$ and $r_1^2 = r_2^2 + r_3^2$ then θ_3 is equal to:

A. $\frac{\pi}{2}$
B. $\frac{\pi}{12}$
C. $\frac{\pi}{6}$
D. $\frac{\pi}{4}$

Answer (A)

Sol.

If a chord of circle of radius R subtends angle θ_i at the centre then locus of the midpoint of this chord is a circle of radius $r_i = R.\cos\left(\frac{\theta_i}{2}\right)$



Given,

$r_1^2 = r_2^2 + r_3^2$

$\Rightarrow cos^2\frac{\theta_1}{2} = cos^2\frac{\theta_2}{2} + cos^2\frac{\theta_3}{2}$

$\Rightarrow cos^2\frac{\pi}{6} = cos^2\frac{\pi}{3} + cos^2\frac{\theta_3}{2}$

$\Rightarrow \frac{3}{4} = \frac{1}{4} + cos^2\frac{\theta_3}{2}$

$\Rightarrow cos^2\frac{\theta_3}{2} = \frac{1}{2}$

$\Rightarrow \cos\frac{\theta_3}{2} = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{\theta_3}{2} = \frac{\pi}{4}$

$\therefore \theta_3 = \frac{\pi}{2}$

14. Let $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3)$, then:

A. $f(0) = f(1) + f(2) + f(3)$
B. $f(3) + 2f(0) = f(2) + f(1)$
C. $2f(0) = f(1) - f(2)$
D. $f(3) - f(1) = 2f(2)$

Answer (B)

Sol.

Let $f(x) = x^3 - Ax^2 + Bx - C$

$\Rightarrow f'(1) = 3 - 2A + B = A$

$\Rightarrow f''(2) = 12 - 2A = B$

$\Rightarrow f'''(3) = 6 = C$

Solving, $f(x) = x^3 - 3x^2 + 6x - 6$

$f(0) = -6, f(1) = -2$

$f(2) = 8 - 12 + 12 - 6 = 2$

$f(3) = 27 - 27 + 18 - 6 = 12$

$\therefore f(3) + 2f(0) = f(2) + f(1)$

15. $\frac{1^3+2^3+3^3+\cdots up\ to\ n\ terms}{1\cdot3+2\cdot5+3\cdot7+\cdots+up\ to\ n\ terms} = \frac{9}{5}$ then n is equal to:

A. 5
B. 7
C. 12
D. 9

Answer (B)

Sol.

$$\frac{\mathbf{1^3+2^3+3^3+\cdots up\ to\ n\ terms}}{\mathbf{1\cdot3+2\cdot5+3\cdot7+\cdots+up\ to\ n\ terms}} =$$

$$\frac{\mathbf{1^3+2^3+3^3+\cdots+n^3}}{\sum \mathbf{n(2n+1)}}$$

$$\Rightarrow \frac{\frac{n^2(n+1)^2}{4}}{2\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}} = \frac{9}{5}$$

$$\Rightarrow \frac{\frac{n(n+1)}{4}\cdot 6}{2(2n+1)+3} = \frac{9}{5}$$

$\Rightarrow 5n^2 - 19n - 30 = 0$

$\Rightarrow 5n^2 - 25n + 6n - 30 = 0$

$\Rightarrow n = 5,\ -\frac{6}{5} \Rightarrow n = 5$

16. If $\left|\boldsymbol{adj}\left(\boldsymbol{adj}(\boldsymbol{adjA})\right)\right| = \mathbf{12^4}$ then $|\boldsymbol{A}^{-1}(\boldsymbol{adjA})|$ equals:(where $\boldsymbol{A}$ is matrix of order $\mathbf{3} \times \mathbf{3}$)

A. $2\sqrt{3}$
B. 1
C. 6
D. 12

Answer (A)

Sol.

$\left|adj\left(adj(adjA)\right)\right| = |A|^{2^3} = |A|^8 = 12^4$

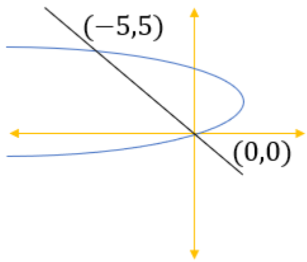
$\therefore |A| = 2\sqrt{3}$

$|A^{-1}adj(A)| = |A^{-1}| \cdot |adj(A)| = \frac{1}{|A|} \cdot |A|^2 = |A| = 2\sqrt{3}$

17. If area bound between $y^2 - 4y = -x$ and $x + y = 0$ is A then $6A$ equals.

Answer (125/6)

Sol.



Point of intersection of $y^2 - 4y = -x$ and $x + y = 0$ is

$$y^2 - 4y = y$$

$$\Rightarrow y^2 - 5y = 0$$

$$\Rightarrow y = 5 \text{ or } y = 0$$

$$\therefore x = -5 \text{ or } x = 0$$

Required Area $= \int_0^5 (4y - y^2) - (-y)\, dy$

$$= \int_0^5 (5y - y^2) dy$$

$$= \left[\frac{5y^2}{2} - \frac{y^3}{3}\right]_0^5$$

$$\Rightarrow \frac{125}{2} - \frac{125}{3} = \frac{125}{6}$$

18. Let $a_1, a_2, \dots, a_6$ be in A.P. such that $a_1 + a_3 = 10$ & mean of $a_1, a_2, \dots, a_6$ is $\frac{19}{2}$. Then $8\sigma^2$ is equal to ______.

Answer (210)

Sol.

$a_1 + a_3 = 10$

$\Rightarrow 2a + 2d = 10$

$$\Rightarrow a + d = 5$$

Also, $\frac{a+(a+d)+\cdots+(a+5d)}{6} = \frac{19}{2}$

$$\Rightarrow 2a + 5d = 19$$

$$\therefore a = 2 \,\&\, d = 3$$

$\therefore$ Given A.P. is 2,5,8,11,14,17

$$\Rightarrow \sigma^2 = \frac{\left(2-\frac{19}{2}\right)^2 + \left(5-\frac{19}{2}\right)^2 + \cdots + \left(17-\frac{19}{2}\right)^2}{6} = 26.25$$

$$\therefore 8\sigma^2 = 8 \times 26.25 = 210$$